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DISPERSION AND WAVE PROPAGATION IN DISCRETE AND CONTINUOUS MODELS FOR GRANULAR MATERIALS

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Abstract—A generalised continuum model for granular media is derived by direct homogenisation of the discrete equations of motion. In contrast to previous works on this topic, continuum concepts such as stress and moment stress are introduced after homogenisation. First, a very simple one-dimensional model is considered and the continuum version for this model is derived by replacing the difference quotients of the discrete model by differential quotients. The dispersion relations of the discrete and the continuous model are derived and compared. Variational boundary conditions for the continuous model are deduced from the stationarity of the corresponding Lagrangian. The three-dimensional case is treated in an essentially similar fashion. The resulting continuum theory is a combination of a Cosserat Continuum and a higher-order deformation gradient continuum. The salient features of the theory are illustrated by means of the dispersion relations for planar wave propagation. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

In most previous works on continuum models for random granular assemblies (Digby, 1981; Mühlhaus and Vardoulakis, 1987; Walton, 1987; Jenkins, 1991; Mühlhaus *et al.*, 1991) certain *a priori* assumptions are made with respect to relationships between the statical and kinematical quantities of the continuum model envisaged and the original discrete system, the granulate. These assumptions are not critical in the case of homogeneous or almost homogeneous deformations, where higher-order deformation gradients or higher-order rotation gradients do not play a role. For strongly inhomogeneous deformations however [e.g. upon shear banding Mühlhaus and Vardoulakis (1987)] the situation is different. In this case higher-order deformation gradients have to be considered leading to a generalised continuum theory of some kind. In an attempt to extend the validity of his standard continuum model to strongly inhomogenous deformation, Jenkins (1991) adopted a nonlocal interpretation of the Cauchy–Love relation (Love, 1927). Vardoulakis and Aifantis (1989) have included gradients of the plastic strain into the dilatancy constraint equation in order to account for the strong spatial nonuniformity of the deformation upon shear banding. Mühlhaus *et al.* (1991) defined average stress and moment-stress tensors by equating the virtual work of a Cosserat Continuum to the corresponding expression of the discrete system. There are many possibilities. The question now is which of the continuum models comes closest to the behaviour of a discrete particulate material. In this paper, continuum relations are derived from the discrete equations of motion by replacing the difference expressions of the discrete model by appropriate differential expressions. In this way any bias towards a particular continuum theory is avoided. Not too surprisingly, the result turns out to be a combination of a Cosserat theory and a strain gradient theory. The relative importance of the nonstandard terms, namely the deformation gradient terms in the constitutive relations, is elucidated by means of the dispersion relation for planar wave propagation. Corresponding dispersion relations for a discrete system of spherical grains have been derived by Walton (1988). The results are compared in Section 3.5. As in Waltons' paper, at the starting point of the present derivation stand the equations of motion for a dense discrete assembly of spherical grains. However in contrast to Walton's approach, we first turn to the homogenisation of the problem. The dispersion relations are then derived for the homogenised model. In agreement with the asymptotic character of the present theory it turns out that the dispersion relations coincide up to the fourth order in the wavenumber. It should be mentioned that Jenkin's (1991) model does not represent a homogenised version of a discrete model and consequently a comparable coincidence cannot be expected in this case. We begin in the next section with an introductory discussion of a strongly simplified model for a one-dimensional granulate.

Throughout the paper we assume that the deformation is infinitesimal, the packing dense enough to ensure solid like behaviour and for simplicity we consider an idealised material consisting of identical spherical grains.

2. ONE-DIMENSIONAL MODELS

2.1. *Formulation*

First we use the Fermi-Pasta-Ulam (FPU) oscillator (Fermi, 1965; Tabor, 1989) as a cartoon for one-dimensional dynamical processes in granular media. We shall later show that the mathematical structure of this simple model is indeed consistent with the one-dimensional version of the three-dimensional model we derive subsequently. The FPU oscillator consists of N equally spaced masspoints, each having mass m, which are connected by nonlinear elastic springs. The translation of the jth masspoint is designated as u_j and we define $\delta_j = u_j - u_{j-1}$. The force–relative displacement $(P_j - \delta_j)$ relation of the FPU chain reads

$$P_j = k\delta_j(1 + \alpha\delta_j), \tag{1}$$

where k and α are constants. Within the context of granular materials one assumes that the masspoints consist of spherical grains which interact through Hertz–Mindlin contacts. In a uniaxial deformation the solution for the normal component of the contact force reads (Love, 1927)

$$P_j = M\left(3\frac{\delta_j}{D}\right)^{3/2}, \quad M = \frac{2\mu D^2}{9\sqrt{3}(1-\nu)}, \tag{2}$$

where μ and ν are the shear modulus and the Poisson ratio of the sphere material, respectively, and D is the sphere diameter. Now we assume that the granular chain is prestressed under the static load P_0 and

$$\delta_{j0} = \delta_0 = \frac{D}{3}\left(\frac{P_0}{M}\right)^{2/3} \tag{3}$$

is the corresponding relative displacements at the grain contacts. The incremental force–relative displacement relationship for small deviations from the equilibrium state (P_0, δ_0) is obtained by Taylor expansion of eqn (2) as

$$\tilde{P}_j = k\tilde{\delta}_j(1 + \alpha\tilde{\delta}_j), \quad k = \frac{9M}{\Delta}\left(3\frac{\delta_0}{D}\right)^{1/2}, \quad \alpha = \frac{1}{4\delta_0}, \tag{4}$$

where $\tilde{P}_j = P_j - P_0$ and $\tilde{\delta}_j = \delta_j - \delta_0$. Terms of higher than second order in δ_j have been neglected. In the following the tildes over $\tilde{P}_j$ and $\tilde{\delta}_j$ are dropped for convenience.

The total elastic energy and the kinetic energy are obtained as

$$W = \tfrac{1}{2}k \sum_{j=1}^{N-1} \delta_{j+1}^2 \left(1 + \frac{2\alpha}{3}\delta_{j+1}\right), \quad T = \tfrac{1}{2}m \sum_{j=1}^{N-1} \dot{u}_j^2 \tag{5}$$

and we assume that $u_0 = u_N = 0$. Insertion of eqn (5) into Lagrange's equations of motion gives

$$m\ddot{u}_j = k(u_{j+1} - 2u_j + u_{j-1})[1 + \alpha(u_{j+1} - u_{j-1})]. \tag{6}$$

Equation (6) agrees up to terms of order δ_j^3 with the equations of motion for a Toda lattice which is integrable. A number of interesting solutions, including solitons, can be found in Toda's (1988) book on nonlinear lattices.

For the derivation of the continuum version of eqn (6) we first replace the discrete coordinate by a continuous coordinate, i.e. $jD \to x$ and we establish the relationship between u_j and $u_{j\pm 1}$ by writing (Toda, 1988)

$$u(x \pm D) = e^{\pm D\frac{\partial}{\partial x}}[u(x)], \tag{7}$$

where the shifting operator is understood as

$$e^{\pm D\frac{\partial}{\partial x}} = 1 \pm \frac{1}{1!}D\frac{\partial}{\partial x} + \frac{1}{2!}D^2\frac{\partial^2}{\partial x^2} \pm \dots. \tag{8}$$

Inserting eqn (7) into eqn (6) gives

$$m\ddot{u} = -2k\left(1 - \cosh D\frac{\partial}{\partial x}\right)u\left[1 + 2\alpha \sinh D\frac{\partial}{\partial x}u\right] \tag{9}$$

and from this, by Taylor expansion

$$m\ddot{u} = kD^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{D^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{D^4}{360}\frac{\partial^6 u}{\partial x^6}\right) + k\beta D^2\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{D^2}{6}\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{D^2}{6}\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4}\right] + O(\beta D^6, D^8) \tag{10}$$

where $\beta = \alpha D$. Note that the $\partial^4 u/\partial x^4$ term in eqn (10) is destabilising† so that in general the $\partial^6 u/\partial x^6$ term has to be included to restore the well-posedness of the differential problem.

We now rephrase eqn (10) in terms of stress σ and strain $\varepsilon = \partial u/\partial x$ with the result:

$$\rho\ddot{u} = \frac{\partial \sigma}{\partial x}, \quad \sigma = \frac{6kv_s}{\pi D}\left\{\varepsilon + \frac{D^2}{12}\frac{\partial^2 \varepsilon}{\partial x^2} + \frac{D^4}{360}\frac{\partial^4 \varepsilon}{\partial x^4} + \beta\left[\varepsilon^2 + \frac{D^2}{12}\left(\frac{\partial \varepsilon}{\partial x}\right)^2 + \frac{D^2}{6}\varepsilon\frac{\partial^2 \varepsilon}{\partial x^2}\right]\right\}, \tag{11}$$

where ρ_g and $V_g = \pi D^3/6$ designate the density of the grain material and the grain volume, respectively, $v_s = V_g/V$ is the solid volume fraction and $\rho = v_s\rho_g$.

The asymptotic version of eqn (11) is the famous Korteweg–de Vries (KdV) equation,§ probably the most thoroughly explored nonlinear partial differential equation which has stable solitary waves (solitons) as solutions [see e.g. Tabor (1989) for details].

† Assuming homogeneous b.c.s the deformation energy expressions in eqn (20) corresponding to the terms $\partial^{2\nu}u$, $\nu = 1, 2, \dots$ in the equations of motion read $(-1)^{\nu+1}(\partial^\nu u)^2$; accordingly the $\nu = 1$ term enters with a positive sign (stabilising), the term with $\nu = 2$, with a negative sign (destablising), etc.

§ For proof, first the dimensionless variables $t' = \omega t$, $\omega = kv_s\sqrt{(\rho V_g)}$; $u' = u/D$ are introduced. Equation (11) becomes (dropping primes) $u_{tt} = u_{xxx} + 2\beta u_x u_{xx} + 1/12u_{xxx} +$ h.o.t. Next we look for an asymptotic solution of the form $y \sim \phi(X, T)$, $X = x - t$ and $T = \beta t$. Thus $\phi_{XT} + \phi_X\phi_{XX} + \delta^2\phi_{XXXX} = 0$, $\delta^2 = 1/(24\beta)$. Setting $u = \phi_x$ yields $u_T + uu_X + \delta^2 u_{xxx} = 0$, which is, within trivial scaling, the reduced form of the KdV equation.

2.2. *Dispersion relations*

Next the dispersion relation of the discrete model eqn (6) is compared with that of the continuous model eqn (9) or its approximation, eqn (10). In the case of the discrete model we are looking for modes of the form $u_j = A_q \exp iq(jD - v_p t)$, $j = 1, 2, \ldots N$, $i = \sqrt{-1}$, q is the wave number and v_p is the phase velocity. By insertion into eqn (6) the dispersion function is obtained as

$$v_p^2 = \frac{12kv_s}{\pi D\rho}\left(\frac{1-\cos qD}{(qD)^2}\right) \simeq \frac{6kv_s}{\pi D\rho}[1 - \tfrac{1}{12}(qD)^2 + \tfrac{1}{360}(qD)^4 - \ldots]. \tag{12}$$

For the derivation of the dispersion function of the continuous model we first write the linear part of eqn (9) in the form

$$\ddot{u} = -\frac{12kv_s}{\pi D^3\rho}\left(1 - \cosh D\frac{\partial}{\partial x}\right)u. \tag{13}$$

Insertion of the eigenmode $u = A_q \exp iq(x - v_p t)$ yields

$$v_p^2 = \frac{12kv_s}{\pi D\rho}\left[\frac{1-\cosh iDq}{(qD)^2}\right] = \frac{12kv_s}{\pi D\rho}\left[\frac{1-\cos Dq}{(qD)^2}\right]. \tag{14}$$

That is, if infinitely many terms are included in the Taylor expansion of the differential operator $\cosh D(\partial/\partial x)$, the dispersion functions of the discrete and the continuous model coincide identically. Otherwise one obtains the corresponding approximation. In case of the linearised version of eqn (10), for instance, we obtain

$$v_p^2 = \frac{6kv_s}{\pi D\rho}[1 - \tfrac{1}{12}(qD)^2 + \tfrac{1}{360}(qD)^4], \tag{15}$$

which are the first three terms of the power series expansion of eqn (14).

2.3. *Variational boundary conditions*

In this section variational boundary conditions for the continuum model are deduced from the stationarity of the corresponding Lagrangian. The Lagrangian density is obtained by replacing the differences in eqn (5) by the corresponding differential expression. The variational principle is written as

$$L \to \text{stat.}, \quad L = \int_{t_0}^{t_1}\int_0^L \mathscr{L}\, dx\, dt \tag{16}$$

and, dropping terms of order βD^4 and D^6, the expression for the Lagrangian density $\mathscr{L}$ is obtained as

$$\mathscr{L} = \tfrac{1}{2}\rho\dot{u}^2 - \frac{6kv_s}{\pi D}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{D^2}{8}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{D^2}{6}\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + \tfrac{1}{3}\beta\left(\frac{\partial u}{\partial x}\right)^3 + \ldots\right). \tag{17}$$

The stationarity of $\mathscr{L}$ requires the vanishing of the functional derivative

$$-\frac{\delta L}{\delta u} = \rho\ddot{u} - \frac{6kv_s}{\pi D}\left(\frac{\partial^2 u}{\partial x^2} + \frac{D^2}{12}\frac{\partial^4 u}{\partial x^4} + \ldots\right) \tag{18}$$

and the vanishing of boundary terms [e.g. Courant and Hilbert (1968), p. 179ff]

$$\frac{6kv_s}{\pi D}\left[\left(\frac{\partial u}{\partial x}+\frac{D^2}{12}\frac{\partial^3 u}{\partial x^3}\right)\delta u\right]_0^L = 0,$$

$$\frac{6kv_s}{\pi D}\left[\left(\frac{D^2}{12}\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial \delta u}{\partial x}\right]_0^L = 0,$$

$$\frac{6kv_s}{\pi D}\left[\left(\frac{D^2}{6}\frac{\partial u}{\partial x}\right)\frac{\partial^2 \delta u}{\partial x^2}\right]_0^L = 0. \tag{19}$$

Obviously at the boundaries either $\partial u/\partial x$ or $\partial^2 u/\partial x^2$ has to be equal to zero in order to avoid ambiguity. In this case the Lagrangian can be written in a more symmetric form as

$$\mathscr{L} = \tfrac{1}{2}\rho\dot{u}^2 - \frac{6kv_s}{\pi D}\left[\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{D^2}{24}\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right]. \tag{20}$$

These are of course standard results of the calculus of variations and they are quoted here for completeness and easy reference. It should be mentioned that in case of the three-dimensional continuum the derivation of independent boundary conditions is less trivial [see Mindlin (1964), Mühlhaus *et al.* (1991)]. The difficulty is that once a quantity is known on a surface, then only the normal derivatives (normal to each surface element) can be varied independently. Strategies to overcome this difficulty are described in the abovementioned references.

3. THREE-DIMENSIONAL CONTINUUM MODEL

3.1. *Formulation*

We consider a three-dimensional assembly of identical, spherical grains. In view of the envisaged continuum formulation, summations over grain contacts occurring in the discrete equations of motion are replaced by a corresponding integral, that is we replace

$$\sum_{\text{contacts}}(\cdot) \to \int_\alpha \mathrm{d}n A(\mathbf{r},\mathbf{n})(\cdot) \tag{21}$$

where

$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{22}$$

is the unit vector from the centre of a sphere to a contact on its surface, θ and ϕ are coordinate angles of a spherical coordinate system with the origin at the sphere (grain) centre, $\mathbf{r}$ is the position vector with respect to a spatially fixed frame of reference and

$$\int_\alpha \mathrm{d}n = \int_0^{2\pi}\int_0^{\pi} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{23}$$

The function $A(\mathbf{n})$ accounts for the orientational distribution of contacts so that $A(\mathbf{r},\mathbf{n})\,\mathrm{d}n$ is the probable number of contacts in the element $\mathrm{d}n$ centred at $\mathbf{n}(\theta,\phi)$. When the distribution of contacts is isotropic and independent of the position then $A(\mathbf{r},\mathbf{n}) = \kappa/4\pi$, where κ is the coordination number, the average number of contacts per grain.

Using the relation (21) the equations of motion can be written as

$$\rho\ddot{\mathbf{u}} = \frac{6v_s}{\pi D^3}\int_\alpha A\mathbf{F}^n\,\mathrm{d}n \quad \text{and}\ \frac{\rho}{10}D^2\ddot{\boldsymbol{\omega}} = \frac{3v_s}{\pi D^2}\int_\alpha A\mathbf{n}\times\mathbf{F}^n\,\mathrm{d}n \tag{24}$$

where $\mathbf{u}(\mathbf{r}, t)$ and $\omega(\mathbf{r}, \mathbf{t})$ are now understood as the expectation values of the translations and rotations taken over all realisations in which the grain centre is originally at $\mathbf{r}$; $\mathbf{F}^{\mathbf{n}}$ is the force exerted by the sphere at the contact at $\mathbf{r}+(D/2)\mathbf{n}$ and as before, v_s designates the solid volume fraction and D is the sphere diameter.

In the following, the expectation values for the motion of grains originally at $\mathbf{r}+D\mathbf{n}$ are assumed as independent of whether or not there is another grain at $\mathbf{r}$. This assumption is usual in connection with dense materials and simplifies the mathematical treatment considerably [see e.g. Willis (1981)].

The force components normal and parallel to the tangential plane of a contact depend upon the relative displacements and rotations between adjacent grains. The relative displacement $\Delta\mathbf{u}^{\mathbf{n}}$ between the contact points of two spheres centred at $\mathbf{r}$ and $\mathbf{r}+D\mathbf{n}$, respectively, reads

$$\Delta\mathbf{u}^{\mathbf{n}} = \mathbf{u}^{\mathbf{n}}-\mathbf{u}+D\mathbf{n}\times\boldsymbol{\omega}+\frac{D}{2}\mathbf{n}\times(\boldsymbol{\omega}^{\mathbf{n}}-\boldsymbol{\omega}) \tag{25}$$

where $\mathbf{u}^{\mathbf{n}} = \mathbf{u}(\mathbf{r}+D\mathbf{n})$. Assuming linear elasticity the $\mathbf{F}-\Delta\mathbf{u}$ relation reads

$$\mathbf{F}^{\mathbf{n}} = \mathbf{K}\Delta\mathbf{u}^{\mathbf{n}}, \quad K_{ij} = (k_n-k_s)n_in_j+k_s\delta_{ij}, \tag{26}$$

where k_n and k_s are the normal and tangential contact stiffnesses and the indices refer to a spatially fixed cartesian coordinate system. To simplify the algebra of the derivation it is assumed from now on that A is independent of position. Then, because a contact is common to two spheres, $A(-\mathbf{n}) = A(\mathbf{n})$. Inserting eqn (26) into eqn (24) the equations of motion are obtained as:

$$\rho\ddot{\mathbf{u}} = \frac{6v_s}{\pi D^3}\int_{\alpha/2} A\mathbf{K}(\mathbf{u}^{\mathbf{n}}-2\mathbf{u}+\mathbf{u}^{-\mathbf{n}})\,dn+\frac{3v_s}{\pi D^2}\int_{\alpha/2} Ak_s\mathbf{n}\times(\boldsymbol{\omega}^{\mathbf{n}}-\boldsymbol{\omega}^{-\mathbf{n}})\,dn, \tag{27}$$

$$\frac{\rho}{10}D^2\ddot{\boldsymbol{\omega}} = \frac{3v_sk_s}{\pi D^2}\int_{\alpha/2} A\mathbf{n}\times(\mathbf{u}^{\mathbf{n}}-\mathbf{u}^{-\mathbf{n}})\,dn+\frac{6v_sk_s}{\pi D}\int_{\alpha/2} A\mathbf{n}\times\mathbf{n}\times\boldsymbol{\omega}\,dn$$
$$+\frac{3v_sk_s}{\pi D}\int_{\alpha/2} A\mathbf{n}\times\mathbf{n}\times(\boldsymbol{\omega}^{\mathbf{n}}-2\boldsymbol{\omega}+\boldsymbol{\omega}^{-\mathbf{n}})\,dn. \tag{28}$$

We have used the symmetry property $A(-\mathbf{n}) = A(\mathbf{n})$ so the integration extends only over half of the solid angle, i.e.

$$\int_{\alpha/2} dn = \int_0^{2\pi}\int_0^{\pi/2}\sin\theta\,d\theta\,d\phi. \tag{29}$$

Next $\mathbf{u}^{\pm\mathbf{n}} = \mathbf{u}(\mathbf{r}\pm D\mathbf{n})$ and $\boldsymbol{\omega}^{\pm\mathbf{n}}$ are replaced by the Taylor expansions

$$u_i^{\pm\mathbf{n}} = u_i\pm\frac{D}{1!}u_{i,j}n_j+\frac{D^2}{2!}u_{i,jk}n_jn_k\pm\ldots, \tag{30}$$

and

$$\omega_i^{\pm\mathbf{n}} = \omega_i\pm\frac{D}{1!}\omega_{i,j}n_j+\frac{D}{2!}\omega_{i,jk}n_jn_k\pm\ldots, \tag{31}$$

where $(\cdot)_{,i} = \partial/\partial x_i(\cdot)$. Neglecting terms of higher than fourth order in D yields

$$u_i^{\mathrm{n}} - 2u_i + u_i^{-\mathrm{n}} \simeq D^2\left(u_{i,jk}n_jn_k + \frac{D^2}{12}u_{i,jklm}n_jn_kn_ln_m\right) \tag{32}$$

and

$$u_i^{\mathrm{n}} - u_i^{-\mathrm{n}} \simeq 2D\left(u_{i,j}n_j + \frac{D^2}{6}u_{i,jk}n_jn_k\right). \tag{33}$$

Analogous expressions are obtained for the ω-differences. What remains is to insert the Taylor expansions into the eqns (27) and (28). The result reads:

$$\rho\ddot{u}_i = \sigma_{ij,j}, \tag{34}$$

$$\sigma_{ij} = \frac{6v_{\mathrm{s}}}{\pi D}\left[(k_{\mathrm{n}}-k_{\mathrm{s}})A_{ijlm}\varepsilon_{lm} + k_{\mathrm{s}}A_{lj}\varepsilon_{il} + k_{\mathrm{s}}A_{lj}(W_{il} - W_{il}^{\mathrm{c}}) + \frac{D^2}{12}[(k_{\mathrm{n}}-k_{\mathrm{s}})A_{ijklmn}\varepsilon_{kl,mn} + k_{\mathrm{s}}A_{ljmn}\varepsilon_{il,mn} + k_{\mathrm{s}}(A_{ljmn}W_{il,mn} - 2A_{ljmn}W_{il,mn}^{\mathrm{c}})]\right], \tag{35}$$

where

$$A_{i_1i_2\ldots i_{2n}} = \int_{\alpha/2} An_{i_1}n_{i_2}\ldots n_{i_{2n}}\,\mathrm{d}n, \tag{36}$$

ε_{ij} and W_{ij} are the symmetric and skew symmetric part of the displacement gradient respectively,

$$W_{ij}^{\mathrm{c}} = -e_{ijk}\omega_k, \tag{37}$$

e_{ijk} with $e_{123} = 1$ is the permutation symbol and the superscribed c (for Cosserat) is used to distinguish the particle spin from the spin W_{ij} of an infinitesimal element $\mathrm{d}V$.

In similar fashion one obtains for eqn (28):

$$\frac{\rho D^2}{10}\ddot{\omega}_i = \frac{6v_{\mathrm{s}}k_{\mathrm{s}}}{\pi D}\left[e_{ijk}A_{lj}(\varepsilon_{kl} + W_{kl} - W_{kl}^{\mathrm{c}}) + e_{ijk}A_{\ell jmn}\frac{D^2}{6}u_{k,lmn} + e_{ijk}e_{klm}A_{jlrs}\frac{D^2}{4}\omega_{m,rs}\right] \tag{38}$$

or

$$\frac{\rho D^2}{10}\ddot{\omega}_i = \mu_{is,s} + e_{ijk}\sigma_{kj}, \tag{39}$$

where the moment stress tensor $\boldsymbol{\mu}$ introduced in this way is obtained as

$$\mu_{is} = \frac{6v_{\mathrm{s}}k_{\mathrm{s}}}{\pi D}\frac{D^2}{12}e_{ijk}[A_{ljms}(u_{k,lm} - W_{kl,m}^{\mathrm{c}})]. \tag{40}$$

Using the standard formula

$$e_{ijk}e_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{41}$$

eqn (40) can be written somewhat more explicitly as

$$\mu_{is} = \frac{6v_{\mathrm{s}}k_{\mathrm{s}}}{\pi D}\frac{D^2}{12}[e_{ijk}A_{ljms}\varepsilon_{kl,m} - A_{inms}(\Omega_{n,m} - \omega_{n,m}) + A_{ms}(\Omega_{i,m} - \omega_{ij,m})], \tag{42}$$

where

$$\Omega_{n,m} = -\tfrac{1}{2} e_{nkl} W_{kl,m}. \tag{43}$$

We conclude this section with the derivation of the Lagrangian density or action density $\mathscr{L}$, corresponding to the eqns (35) and (38). We have

$$\mathscr{L} = \mathscr{T} - \mathscr{W}, \tag{44}$$

where in the present case

$$\mathscr{T}(\mathbf{r}, t) = \tfrac{1}{2}\rho\left(\dot{u}_i^2 + \frac{D^2}{10}\dot{\omega}_i^2\right) \tag{45}$$

and

$$\mathscr{W}(\mathbf{r}, t) = \mathscr{W}(u_{i,j}, u_{i,jk}, W^c_{ij}, W^c_{ij,k}). \tag{46}$$

For the identification of $\mathscr{W}$, first the right-hand sides of eqns (34) and (39) are multiplied by $\dot{u}_i\,dt$ and $\dot{\omega}_i\,dt$, respectively. Integration over the space–time domain of the deformation under consideration and successive application of Gauss' theorem, yields:

$$\mathscr{W} = \frac{1}{2}\frac{6v_s}{\pi D}\left[(k_n - k_s)A_{ijlm}\varepsilon_{ij}\varepsilon_{lm} + k_s A_{lj}(u_{i,l} - W^c_{il})(u_{i,j} - W^c_{ij}) - \frac{D^2}{12}(k_n - k_s)A_{ijklmn}\varepsilon_{kl,m}\varepsilon_{ij,n}\right.$$
$$\left. - \frac{D^2}{12}k_s A_{ljmn}(u_{i,lm} - W^c_{il,m})(u_{i,jn} - W^c_{ij,n}) + \frac{D^2}{6}k_s A_{ljmn} W^c_{il,m}(u_{i,jn} - W^c_{ij,n})\right]. \tag{47}$$

The boundary integrals appearing during the derivation have been assumed to vanish. As in the one-dimensional case treated in Section 2, the second-order deformation gradients in $\mathscr{W}$ destabilise the deformation so that in numerical analyses third-order gradients have to be included into the expansion in order to stabilise the solution at a finite wave length. The second-order and deformation gradients are destabilising, third-order gradients are stabilising, fourth-order gradients are destabilising again and so forth. At first inspection of eqn (47) one could get the impression that the spins $\mathbf{W}$ and $\mathbf{W}^c$ would enter into the complete model, of which eqn (47) is an approximation, in the form of the relative spins $\mathbf{W} - \mathbf{W}^c$ only. As one easily verifies by Taylor expanding $\mathbf{u}^n$ and $\mathbf{W}^n$ in eqn (25), the prominent appearance of the $\mathbf{W} - \mathbf{W}^c$ in the energy of deformation expression is just a feature of the present order of approximation. We discuss a number of special cases for which the coefficients are evaluated in detail. First the standard continuum is considered where higher-order deformation gradients are neglected.

3.2. *Standard continuum*

In order to obtain explicit expressions for the elasticities we have to make an assumption concerning the form of the contact distribution function $A(\mathbf{n})$. For orthotropic depositional anisotropy $A(\mathbf{n})$ can be expressed in terms of a symmetric, traceless tensor $\mathbf{A}$ (Kanatani, 1984; Cowin, 1985):

$$A(\mathbf{n}) = \frac{k}{4\pi}(1 + A_{ij}n_i n_j). \tag{48}$$

For transversely isotropic materials, $\mathbf{A}$ may be expressed in terms of the unit vector $\mathbf{h}$ in the direction of the axis of anisotropy and the strength of the anisotropy:

$$A_{ij} = -a(\delta_{ij} - 3h_i h_j) \tag{49}$$

where $0 \leqslant a \leqslant 1$. Then

$$A(\mathbf{n}) = \frac{\kappa}{4\pi}[(1-a)+3a(h_i n_i)^2]. \tag{50}$$

Next the components of the tensors $A_{i_1 i_2 \ldots i_{2n}}$ are to be evaluated. Note that according to the definition (36) all integrations over the angular domain of (θ, ϕ) extend over half of the solid angle only. Integrals that facilitate this calculation are (Kanatani, 1984; Jenkins, 1991):

$$I_{i_1 i_2 \ldots i_{2n}} = \int_{\pi/2} n_{i_1} n_{i_2} \ldots n_{i_{2n}} \, \mathrm{d}n = \frac{1}{2}\frac{4\pi}{2n+1}, \quad \text{for } i_1 = i_2 = \ldots i_{2n}, \tag{51}$$

and

$$2I_{ij} = \frac{4\pi}{3}\delta_{ij}, \quad 2I_{ijkl} = \frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

$$I_{ijklmn} = \tfrac{1}{7}(\delta_{in}I_{jklm}+\delta_{jn}I_{klmi}+\delta_{kn}I_{lmij}+\delta_{ln}I_{mijk}+\delta_{mn}I_{ijkl}). \tag{52}$$

The result for the stress–deformation relation (35) reads:

$$\sigma_{ij} = \frac{v_s \kappa}{5\pi D}\{\tfrac{1}{7}(k_n - k_s)[(7-4a)(2\varepsilon_{ij}+\varepsilon_{kk}\delta_{ij})+18a(\varepsilon_{kk}h_k h_l \delta_{ij}+\varepsilon_{kk}h_i h_j$$

$$+2h_i\varepsilon_{jk}h_k+2h_j\varepsilon_{ik}h_k)]+k_s[(5-2a)(u_{i,j}-W^c_{ij})+6ah_l h_j(u_{i,l}-W^c_{il})]\}. \tag{53}$$

When the deformation is homogeneous and body couples are absent, the stress must be symmetric. Symmetry of σ_{ij} requires that

$$3ah_l(\varepsilon_{il}h_j-\varepsilon_{jl}h_i)+3ah_l[(W_{il}-W^c_{il})h_j-(W_{jl}-W^c_{jl})h_i]+(5-2a)(W_{ij}-W^c_{ij}) = 0. \tag{54}$$

Solving for W^c_{ij} yields

$$W^c_{ij} = W_{ij}+\frac{3a}{5+a}(\varepsilon_{il}h_j-\varepsilon_{jl}h_i)h_l, \tag{55}$$

where the identity

$$h_l[\Omega_{il}h_j-\Omega_{jl}h_i] = \Omega_{ij}, \quad \Omega_{ij} = -\Omega_{ji}, \quad h_i h_i = 1 \tag{56}$$

has been used.

For $a = 0$ the material is isotropic. The Lamé coefficients μ and λ are related to the contact stiffnesses as

$$\mu = \frac{v_s\kappa}{5\pi D}(k_n+\tfrac{3}{2}k_s) \quad \text{and} \quad \lambda = \frac{v_s\kappa}{5\pi D}(k_n-k_s). \tag{57}$$

Combination of the relations (57) yields $k_s = 2\pi D(\mu-\lambda)/(v_s\kappa)$, and, since $k_s \geqslant 0$ it follows that the Poisson ratio of a random packing of spherical grains has to be less than or equal to 0.25 (Walton, 1987). Jenkins (1988) has derived relations between k_n, k_s and the moduli of the grain material for an infinitesimal deviation from a homogeneously and isotropically prestrained ground state. The result reads

$$k_n = \frac{9}{2D} M \Delta^{1/2}, \quad k_s = 2\frac{(1-\nu_g)}{(2-\nu_g)} k_n, \quad \text{with } M = \frac{2}{9\sqrt{3}} \frac{\mu_g D^2}{(1-\nu_g)}, \tag{58}$$

where ν_g and μ_g are the Poisson ratio and the shear modulus, respectively, of the grain material and Δ is the magnitude of the volume strain of the ground state.

3.3. *Isotropic fabric, general case*

Inserting the relations (49)–(52) into the constitutive relationships (35) and (40) yields, for $a = 0$ (isotropic contact distribution):

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} + 2(\mu-\lambda)(W_{ij} - W^c_{ij}) + \frac{D^2}{12}\{\tfrac{6}{7}\lambda(\varepsilon_{ij,kk} + \varepsilon_{kk,ij} + \tfrac{1}{2}\varepsilon_{kk,mm}\delta_{ij})$$
$$+\tfrac{6}{5}(\mu-\lambda)\varepsilon_{ij,kk} + \tfrac{6}{5}(\mu-\lambda)(W_{ij,kk} - W^c_{ij,kk}) - \tfrac{8}{5}(\mu-\lambda)W^c_{im,mj} + \tfrac{2}{5}(\mu-\lambda)W^c_{ij,kk}\} \tag{59}$$

and

$$\mu_{is} = \tfrac{2}{5}(\mu-\lambda)\frac{D^2}{12} e_{ijk}(\gamma_{kj,s} + \gamma_{km,m}\delta_{js} + \gamma_{ks,j}), \tag{60}$$

where γ_{ij} designates the relative deformation

$$\gamma_{ij} = u_{i,j} - W^c_{ij}. \tag{61}$$

We note that, as a consequence of the number of terms included in the Taylor expansions eqns (30) and (31), the moment stresses μ_{is} depend on the gradients of the relative deformation tensor γ_{ij} rather than on the gradients of W^c_{ij} alone as in the classical Cosserat theory [see Günther (1958), Schäfer (1967) for reviews on this subject]. As in the Cosserat models for granular media [e.g. Mühlhaus *et al.* (1991)] it is the grain diameter which provides an intrinsic length scale which becomes crucial upon strain localisation, in boundary layers or in high frequency wave-propagation and related phenomena.

It should be mentioned that the structural dependence of μ_{is} on γ_{rs} in eqn (60) is identical to the one derived by Jenkins (1991) in an entirely different way. The bending stiffness differs by a factor of one-half from the corresponding value in Jenkin's paper. A more significant difference is that in Jenkin's model the stress tensor depends exclusively on the relative deformation γ_{ij} and its gradients whereas in the present theory (59) depends on the second gradients of W^c_{ij} as well.

We conclude this section with a brief description of Jenkin's model. Jenkins (1991) considers the eccentricity of the contact forces with respect to the grain centres by writing the Cauchy–Love relation (Love, 1927) in the form

$$\sigma_{ij} = \frac{D}{2}\int_\alpha \int_{-1/2}^{1/2} f_i(\mathbf{r}_c, \mathbf{n}) n_j \, d\xi \, dn, \quad f_i = m(\mathbf{r}_c, \mathbf{n}) F_i(\mathbf{r}_c, \mathbf{n}) \tag{62}$$

where $\mathbf{r}_c = \mathbf{r} + \xi\mathbf{s}, m$ is the particle number density and $\mathbf{s} = D\mathbf{n}$. The corresponding expression for the moment stress is obtained by considering the moment of the contact force about $\mathbf{r}$:

$$\mu_{ij} = \frac{D}{2} e_{ikl} \int_\alpha \int_{-1/2}^{1/2} \xi s_k f_k n_l \, d\xi \, dn \tag{63}$$

So far this is a nonlocal model and the corresponding gradient model is obtained by expanding $\mathbf{f}$ into a Taylor series about $\mathbf{r}$. The model (62) and (63) allows, in principle, for spatial variability of the particle number density. In the derivation of eqns (59) and (60) it was assumed that the solid volume fraction is constant, which for spherical grains of equal radius implies that the particle density is constant. The assumption has been made to enable explicit results and can be dropped in connection with general considerations.

3.4. *Dispersion relations*

First the equations of motion are specialised to one-dimensional deformations, deformations where the fields depend on one coordinate only. Let $x = x_1$ be this coordinate and for deformations in the (x_1, x_2) plane we have

$$\mathbf{u} = u_1(x)\mathbf{e}_1 + u_2(x)\mathbf{e}_2, \quad \boldsymbol{\omega} = \omega_3\mathbf{e}_3. \tag{64}$$

Inserting eqn (64) into the equations of motion, gives for isotropic fabric:

$$c_{0L}^2\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{d^2}{12}\frac{\partial^4 u_1}{\partial x^4}\right) = \ddot{u}_1, \tag{65}$$

$$d^2 = \frac{3}{35}\frac{14\mu + 11\lambda}{2\mu + \lambda}D^2, \quad c_{0L}^2 = \frac{1}{\rho}(2\mu + \lambda) \tag{66}$$

for the longitudinal wave along the x axis, and

$$2(\mu-\lambda)\left(\frac{\partial u_2}{\partial x} + \frac{D^2}{12}\frac{6}{5}\frac{\partial^3 u_2}{\partial x^3}\right) - 4(\mu-\lambda)\left(\omega_3 + \frac{D^2}{12}\frac{6}{5}\frac{\partial^2 \omega_3}{\partial x^2}\right) = \frac{\rho}{10}D^2\ddot{\omega}_3, \tag{67}$$

$$-2(\mu-\lambda)\left(\frac{\partial \omega_3}{\partial x} + \frac{D^2}{12}\frac{6}{5}\frac{\partial^3 \omega_3}{\partial x^3}\right) + (2\mu-\lambda)\frac{\partial^2 u_2}{\partial x^2} + \frac{D^2}{12}\left(\frac{6}{5}\mu - \frac{27}{35}\lambda\right)\frac{\partial^4 u_2}{\partial x^4} = \rho\ddot{u}_2, \tag{68}$$

for the microrotation and shear waves, respectively. The analogy between eqn (65) and the linear part of the fourth-order approximation of eqn (9) is obvious, so there is little need for further discussion at this point. It should be mentioned, however, that the influence of the gradient term is strongest if $\mu = \lambda$, that is, if $k_s = 0$. In this case $d = \sqrt{\frac{5}{7}}$. For $D = 0$ one obtains the longitudinal and shear wave equations of the classical continuum, as it must be. Note that in this case $\partial u_2/\partial x = 2\omega_3$ [eqn (68)] and c_{0L} designates the longitudinal wave speed of the classical continuum (which is obtained here in the limit for infinitely long wave length).

For the derivation of the dispersion functions for the rotational and shear waves we consider travelling wave solutions of the form

$$\begin{bmatrix} u_2 \\ \omega_3 \end{bmatrix} = \exp i(qx - \omega t)\begin{bmatrix} U_2 \\ i\Omega_3 \end{bmatrix}, \tag{69}$$

where $i = \sqrt{-1}$ and q is the wave number. Inserting eqn (69) into eqns (67) and (68) yields the homogeneous system of equations

$$\begin{bmatrix} A_{uu} & A_{u\omega} \\ A_{\omega u} & A_{\omega\omega} \end{bmatrix}\begin{bmatrix} U_2 \\ D\Omega_3 \end{bmatrix} = 0, \tag{70}$$

where

$$A_{uu} = \left(2-\frac{\lambda}{\mu}\right)\gamma^2 - \frac{1}{12}\left(\frac{6}{5}-\frac{27}{35}\frac{\lambda}{\mu}\right)\gamma^4 - \tilde{\omega}^2, \tag{71}$$

$$A_{\omega\omega} = 4\left(1-\frac{\lambda}{\mu}\right)(1-\tfrac{1}{10}\gamma^2) - \tfrac{1}{10}\tilde{\omega}^2, \tag{72}$$

$$A_{\omega u} = A_{u\omega} = 2\left(1-\frac{\lambda}{\mu}\right)(1-\tfrac{1}{10}\gamma^2)\gamma, \tag{73}$$

and

$$c_{0S} = \sqrt{\frac{\mu}{\rho}}, \quad \gamma = qD \quad \text{and} \quad \tilde{\omega} = \frac{D}{c_{0S}}\omega. \tag{74}$$

Equation (70) has nontrivial solutions if

$$\tilde{\omega}_{1/2} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}, \tag{75}$$

where

$$a = \frac{1}{10}, \quad b = -4\left(1-\frac{\lambda}{\mu}\right) + \frac{1}{10}\left(2-3\frac{\lambda}{\mu}\right)\gamma^2 + \frac{1}{120}\left(\frac{6}{5}-\frac{27}{35}\frac{\lambda}{\mu}\right)\gamma^4,$$

$$c = 4\left(1-\frac{\lambda}{\mu}\right)\gamma^2 - \frac{4}{12}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{6}{5}+\frac{3}{7}\frac{\lambda}{\mu}\right)\gamma^4. \tag{76}$$

By expansion of eqn (75) in powers of γ, the leading terms are obtained as

$$\tilde{\omega}_1 = \frac{4\left(1-\dfrac{\lambda}{\mu}\right)}{1/10}(1-\tfrac{3}{40}\gamma^2), \quad \text{and}\, \tilde{\omega}_2 = \gamma^2. \tag{77}$$

By inspection of the amplitude ratio one finds that the plus sign in eqn (75) corresponds to a microrotation wave [that is $D\Omega_3/(\gamma U_2) \gg 1$ within range of wave numbers considered here], a wave type which does not exist in a standard continuum. At least in the range $\gamma \leqslant 1$ the minus sign clearly corresponds to a conventional shear wave. The amplitude ratios as functions of γ and the ratio λ/μ are represented in Fig. 4. It should be mentioned that the existence of microrotation waves is typical for continua with extra degrees of freedom such as the Cosserat Continuum (Suhubi and Eringen, 1964; Sluys, 1992) and Mindlin's (1964) generalisation of it. The present theory is an approximation of the actual behaviour of a granular material. One cannot expect the dispersion relations (75) to yield reasonable values over the whole range of γ. The maximum value of $\gamma(= 1.6)$ in the figures, corresponds to a grain diameter, wave length ratio of about 1/4. The frequency, phase and group velocity functions as functions of the dimensionless wave number γ are represented in Figs 1–3. The group velocity, the velocity at which the envelope of a group of harmonics travels through a medium, has to be positive for physical reasons. In case of the microrotation wave [Fig. 3(b)], positive wave speeds exist only for a limited range of the material parameters, namely for λ/μ close to unity, which is equivalent to $k_s/k_n \ll 1$. For the experimental proof of the existence of such microrotation waves this means that one has to concentrate on the perhaps somewhat academic case of smooth spheres or grains. The situation might change somewhat if even higher deformation gradients are included.

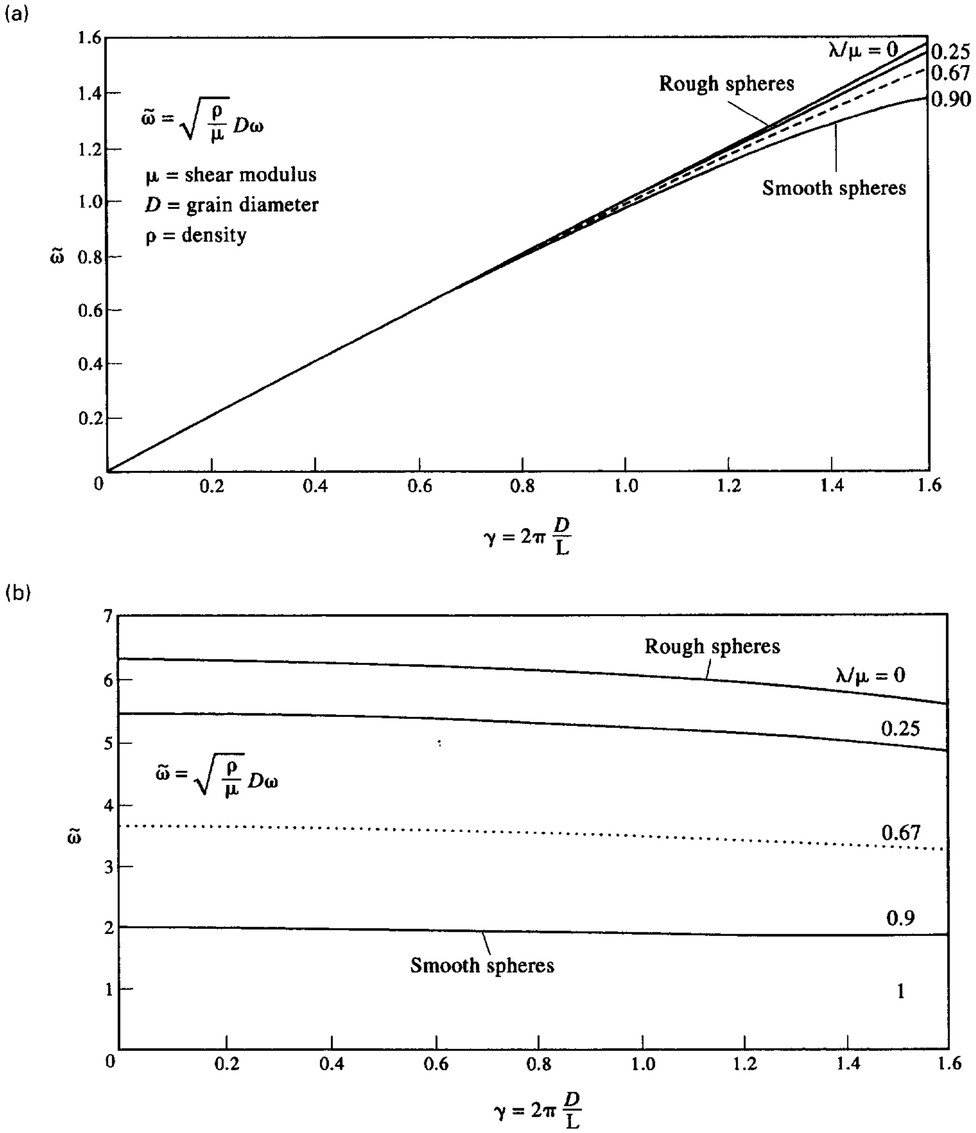


Fig. 1. Angular frequency as function of the dimensionless wave number $\gamma = 2\pi(D/\mathscr{L})$. (a) Shear wave. (b) Microrotation wave.

A special case occurs if the spheres are ideally smooth so that $k_s = 0$ and accordingly $\lambda = \mu$. One then expects an instability which, as a matter of fact, is reproduced by the model. From eqns (75) and (76) it follows for $\lambda = \mu$:

$$\tilde{\omega}_2 = 0 \quad \text{with}\, D\Omega_3/U_2 = \infty \tag{78}$$

and

$$\tilde{\omega}_1 = \gamma^2(1-\tfrac{1}{28}\gamma^2) \quad \text{with}\, D\Omega_3/U_2 = 0. \tag{79}$$

We conclude this section with a comparison of the present results with an analysis carried out by Walton (1988). Invoking statistical considerations Walton derived dispersion relations for various forms of uniaxial wave propopation in a **discrete** granulate. As in the present section, Walton assumes a statistically uniform contact distribution, so one would expect coincidence of the results up to the fourth order in the wave number γ. The general form of the dispersion relations of the discrete model is usually very involved. Therefore

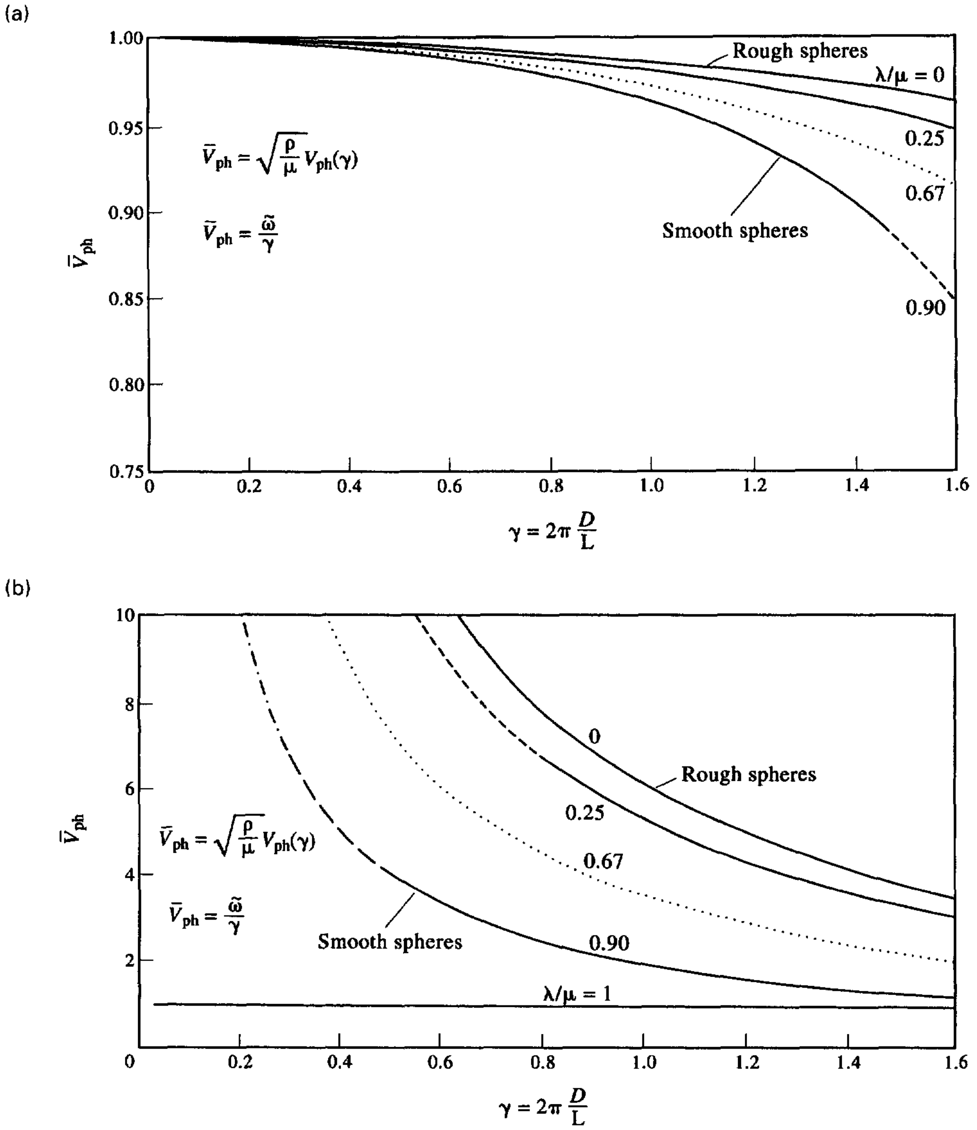


Fig. 2. Phase velocities. (a) Shear wave. (b) Microrotation wave.

we concentrate on some of the more tractable and yet representative cases. Assuming smooth spheres (i.e. $k_s = 0$) Walton's result [eqn. (3.37)] reads:

$$\tilde{\omega}_{1_{\text{Walton}}} = 30\left(\frac{1}{3} + \frac{\cos\gamma}{\gamma^2} - \frac{\sin\gamma}{\gamma^3}\right). \tag{80}$$

Expanding eqn (80) into a Taylor series gives

$$\tilde{\omega}_{1_{\text{Walton}}} = \gamma^2 - \tfrac{1}{28}\gamma^4 + \tfrac{1}{1512}\gamma^6 - o(\gamma^8). \tag{81}$$

Equation (81) coincides up to the fourth order with the present result (79).

For the longitudinal wave eqns (65) and (66), the dispersion relation, are obtained as

$$\tilde{\omega}_L^2 = \gamma^2 - \frac{1}{140}\frac{14\mu + 11\lambda}{2\mu + \lambda}\gamma^4 \tag{82}$$

or, using the relation

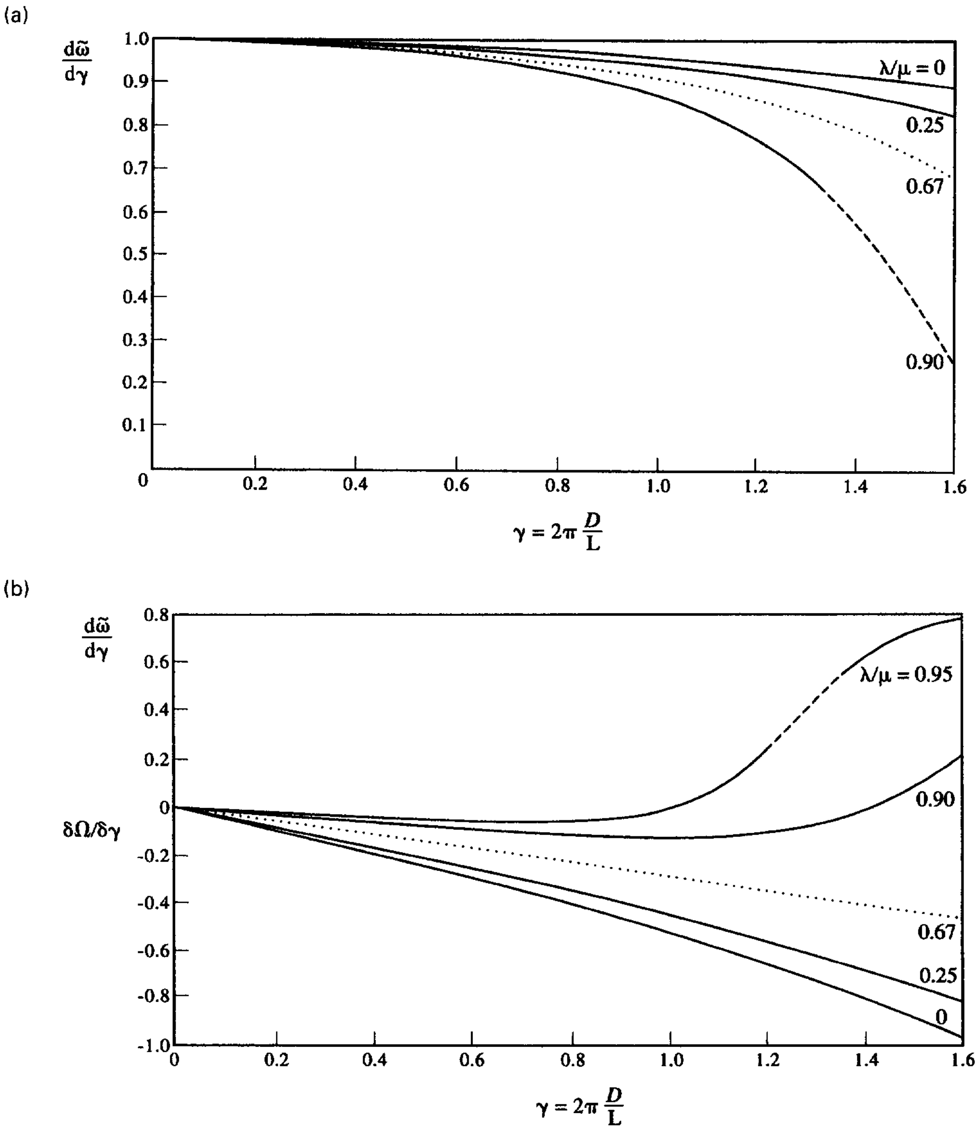


Fig. 3. Group velocities. (a) Shear wave. (b) Microrotation.

$$\frac{\lambda}{\mu} = \frac{k_n - k_s}{k_n + \frac{3}{2}k_s} = \frac{v_g}{5 - 4v_g} \tag{83}$$

$$\tilde{\omega}_L^2 = \gamma^2 - \frac{1}{28}\frac{14 - 9v_g}{10 - 7v_g}\gamma^4 \tag{84}$$

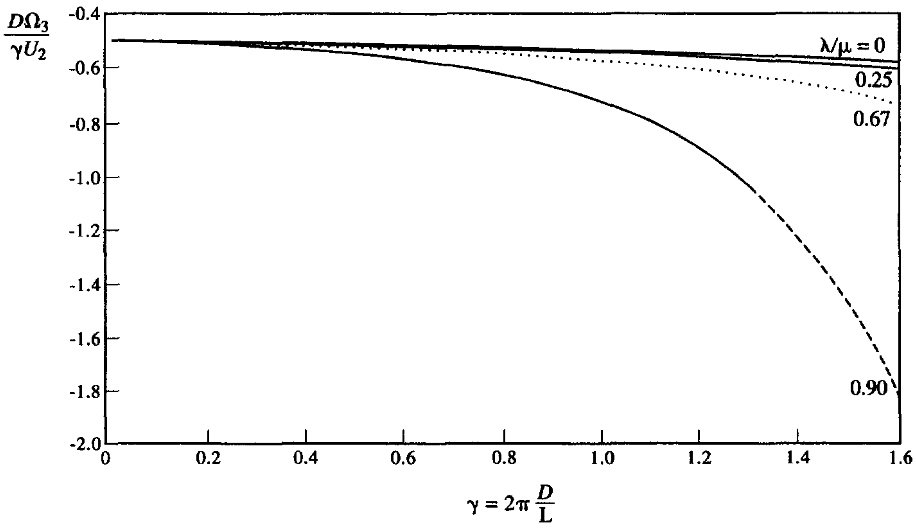
$$\tilde{\omega}_L = \frac{D}{c_{0L}}\omega. \tag{85}$$

The corresponding results by Walton reads

$$\tilde{\omega}_{L_{\text{Walton}}} = \left(\frac{1}{6} + \frac{C}{20B}\right)^{-1}\left[1 - \frac{\sin\gamma}{\gamma} + \frac{C}{2B}\left(\frac{1}{3} - \frac{\sin\gamma}{\gamma} - \frac{2\cos\gamma}{\gamma^2} + \frac{2\cos\gamma}{\gamma^3}\right)\right] \tag{86}$$

where $C/B = v_g/(1 - v_g)$. Again, by developing eqn (86) into a Taylor series one shows that

(a)



(b)

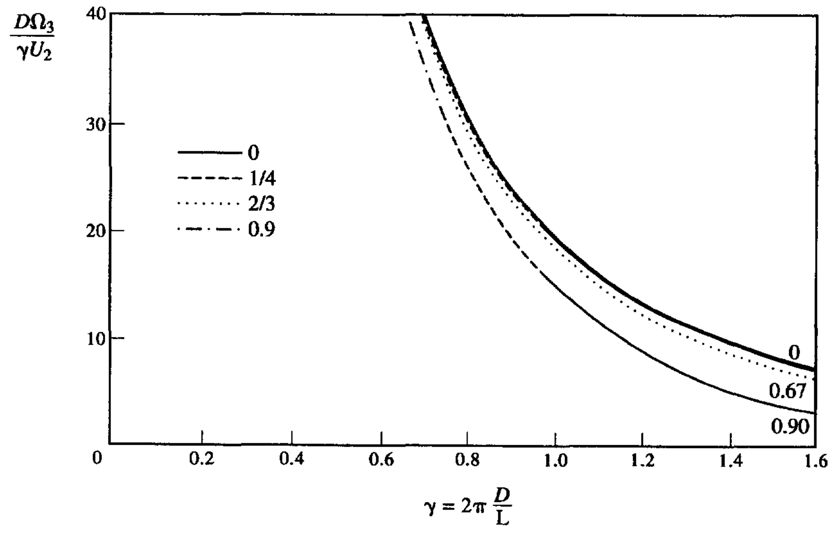


Fig. 4. Amplitude ratios (a) Shear wave. (b) Microrotation.

the results of eqn (86) for the discrete model and eqn (84) coincide exactly up to the fourth order in γ.

4. CONCLUSION

First we have studied a very simple one-dimensional model for the propagation of longitudinal waves in a granular medium. From the discrete version of the model we have derived a continuum model by replacing the difference quotients by differential quotients or Taylor expansions, respectively. We have shown that the dispersion functions of the two models coincide identically if, in the limit, infinitely many terms are included in the Taylor expansion. For isotropic materials odd order derivatives cancel during the homogenisation. It turns out that second-order strain derivatives act as destabilisers. Fourth-order derivatives are stabilisers again and so forth.

We have followed the proposition frequently made in the literature that the grains interact through Hertz–Mindlin contacts [e.g. Walton (1987), Jenkins (1988)]. The Hertz–Mindlin contact stiffnesses are nonlinear functions of the relative displacement between the point of contact and the grain centre. To facilitate the analytical treatment of the model we have expanded the stiffness relations into a Taylor series about a homogeneously prestressed ground state. Truncation of the Taylor expansion after the quadratic term and retaining derivatives of up to the fourth order in the spatial expansion yields the famous Korteweg–de Vries equation, the archetypical equation of soliton dynamics. Next, we have considered the three-dimensional case. The starting point for the derivation of the continuous model are Newton's equations of motion. Similar to the situation in a Cosserat Continuum the equations of motion contain an additional independent, rotational degree of freedom, the grain rotation. However, different from the Cosserat and Mindlin† (1964) Continua, a consequent approximation scaled by powers of the grain diameter requires the inclusion of higher order displacement gradients as well. If higher-order displacement and rotation gradients are neglected, then the grain spin becomes an internal variable. The evolution equation for the grain spin is obtained from the moment equilibrium condition. For isotropic granulates it follows that the grain spin has to be equal to the nonsymmetric part of the displacement gradient. In general, however, for anisotropic fabrics, the relation is less trivial. Finally we have evaluated the dispersion relations for the propagation of longitudinal and shear waves. As expected, the longitudinal wave equation is formally identical to the one of the simplified model discussed in Section 2. The effective stiffness and also the effective grain size however are smaller for the three-dimensional model. In both cases the moduli are expressed in terms of contact stiffness, solid volume fraction and coordination number (average number of contacts per grain) so that a comparison is actually possible. It should be mentioned that for smooth grains (vanishing shear stiffness) the effective grain diameters are equal to the physical grain diameters.

For the shear waves, two dispersion relations are obtained. In the one case the main carrier of the energy is the displacement and in the other case the main carrier is the grain rotation. The latter wave type, which does not exist in standard continua, is typical for micropolar theories of all kinds.

In the present model all parameters of the dispersion relations are determined in the sense that they can be expressed in terms of solid volume fraction, normal contact stiffness, coordination numbers and average grain diameter. It should therefore be possible, at least in principle, to verify, falsify or determine a range of validity of the model by careful measurements of the various wave speeds either in physical experiments or, for a start, by means of discrete element simulations. We wish to emphasise that here the situation is different from previous developments in this area where it was suggested to calibrate certain nonstandard moduli by means of wave speed measurements. In the latter case the verification of the model would require an additional experiment.

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† In his 1964 paper, Mindlin introduces a continuum in which the deformation energy depends on the strain ε_{ij}, the relative deformation $\gamma_{ij} = u_{i,j} - \psi_{ij}$ where ψ_{ij} is the microdisplacement gradient, and the microdeformation gradient $\psi_{ij,k}$. When $\psi_{ij} = -\psi_{ji}$ Mindlin's continuum reduces to a Cosserat Continuum. In the present paper the analogon to ψ_{ij} is the granular spin W^c_{ij}.

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